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Turing-Hopf instabilities through a combination of diffusion, advection, and finite size effects

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We show that in a reaction diffusion system on a two-dimensional substrate with advection in the confined direction, the drift (advection) induced instability occurs through a Hopf bifurcation, which can become a double Hopf bifurcation. The box size in the direction of the drift is a vital parameter. Our analysis involves reduction to a low dimensional dynamical system and constructing amplitude equations. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4859259]

I. INTRODUCTION

Pattern formation in two reacting and diffusing systems was first studied by Turing.[1,2] Turing argued that if the diffusion coefficients of the two species were widely different and one of the species was autocatalytic with the other inhibiting its growth, then the steady homogeneous state will be unstable against the formation of a patterned state. The instability could also set in with a time dependence if the reactive part has an oscillatory feature. Turing patterns have been a very important aspect of the study of non-linear systems.[3–10] Decades later it was found by Rovinsky and Menzinger[11–13] that pattern formation could occur even if the two diffusion coefficients were nearly equal provided there was an external electric field and the diffusing species were charged giving rise to a gradient coupling (an advective term). The pattern formed in this case would be travelling waves as opposed to the stationary patterns of Turing. Recently there has been some extension of the Turing work in some unexpected directions[14–21] covering the effect of an electric field on charged diffusing species and the effect of an advective drift on neutral species. The latter is the situation when the medium has a velocity in a definite direction. While charged reactants with an applied electric field is the simplest realization, we can also envisage a reaction occurring in a container where a fluid mixture is forced in from the left at a constant rate and evacuated at the same rate from the right. This would be the process in a chemical reactor. A more complicated situation would be an open flame but there the wind velocity (drift) could be random. Riaz *et al.* have shown numerically[22,23] that the Turing pattern for charged species could be changed drastically by an applied electric field.[24] In a very recent work the effect of turbulent diffusion on the pattern formation has been considered. In this paper we revisit the problem with a drift to find the as yet overlooked fact that the drift induced instability for nearly equal diffusivities can actually proceed via a double Hopf bifurcation. It is imperative to have a finite sized system for this, so that a homogeneous state is never an option and diffusion can actually induce stability.

The double Hopf bifurcation, also known as a Hopf–Hopf bifurcation is the bifurcation of a fixed point in a two parameter family of autonomous ordinary differential equations in which at the bifurcation point there are two pairs of purely imaginary eigenvalues. This bifurcation lies at the transversal intersection of two curves of Hopf bifurcation. If we consider the dynamics of a complex variable z, then near a Hopf bifurcation point the dynamics of z is governed by the canonical form $\dot{z} = (a + ib)z - z(|z|)^2$. The fixed point $z = 0$ is stable for $a < 0$ and loses stability at $a = 0$. For $a > 0$, a limit cycle of radius $\sqrt{a}$ and frequency b is stabilized so long as the sign in front of the second term in the evolution equation is negative. If the system is described by two complex variables z_1 and z_2 with the evolutions given by (a, b, c, d, f, and g are real constants)

$$\dot{z_1} = (a + ib)z_1 - z_1[|z_1|^2 + f|z_2|^2], \tag{1}$$

$$\dot{z_2} = (c + id)z_2 - z_2[|z_2|^2 + g|z_1|^2], \tag{2}$$

then the stable fixed point (0,0) undergoes a standard Hopf bifurcation if $a = 0$ with $c < 0$ or if $c = 0$ with $a < 0$. If it so happens that there is an underlying parameter which causes $a = c = 0$ at the same point in parameter space then we have an instability which can become a double Hopf bifurcation. In what follows we will show that the drift term in the reaction diffusion system is exactly such a parameter and at a critical value of it there is the possibility of a double Hopf instability. Double Hopf bifurcations have usually been associated with oscillators with time delays[25–27] and some swirling flows in hydrodynamics.[28,29] Our point is that the drift induced pattern formation in reaction diffusion systems provides a natural setting for the double Hopf bifurcation.

Here we work with a two-dimensional set up as in the work of Riaz *et al.* We take the system to be unbounded in the y-direction and $\pm L$ in x-direction. The boundary condition is that the current normal to the plates has to vanish at the boundaries. This means the x-derivatives of the different concentration fields will vanish at the boundaries. The electric field is taken to be in the x-direction which leads to a drift in that direction. The existence of the plates in the x-direction is

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vital to keep the system bounded. The plates also play the very important role of fixing the wave number in the x-direction which ensures that the homogeneous state is never an option. In the absence of the constraint there will be an overall selection mechanism for the wave numbers ($k = k_1 + k_2$) but the individual components are not uniquely determined. In the presence of the plates, k_1 is fixed at $\frac{\pi}{2L}$.

II. THE DYNAMICAL SYSTEM AND LINEAR STABILITY ANALYSIS

The general reaction diffusion problem for two species $A(x, y, t)$ ad $B(x, y, t)$ can be modelled by the evolution equations:

$$\dot{A} = D\nabla^2 A + f_2 E D\frac{\partial A}{\partial x} + f(A, B), \tag{3}$$

$$\dot{B} = \nabla^2 B + f_1 E\frac{\partial B}{\partial x} + g(A, B), \tag{4}$$

where f_1 and f_2 are dimensionless numbers which describe the coupling strength of the applied field to the species "A" and "B." The functions $f(A, B)$ and $g(A, B)$ are reactions rates and have zeroes at $A = A_0$ and $B = B_0$ which correspond to equilibrium points for the reactions. In general it is very difficult to analytically go beyond a linear stability analysis. To account for the effect of the nonlinear terms, one needs to carry out a numerical analysis and even then solving a pair of nonlinear coupled partial differential equations (PDE) without any prior insight can be needlessly difficult. It is with this simplification in mind that one carries out a Galerkin truncation of PDEs. In this procedure a set of PDEs can be reduced to a low dimensional dynamical system – a set of coupled ordinary differential equations (ODE). The technique involves expanding the time and space dependent fields of the PDEs in a relevant complete set of spatial modes with time dependent coefficients. Inserting the expansion in a PDE and equating the coefficient of the same mode on either side yield an ODE. The dimension of the dynamical system (the number of ODEs) will be the same as the number of modes that we want to keep from among the complete set. This is a technique that has been used extensively for hydrodynamic problems but to our knowledge not very often for reaction diffusion systems. We would like to implement this scheme here and as a first step towards that expand the variables $A(x, t)$ and $B(x, t)$ about their equilibrium values A_0 and B_0 as $U = A - A_0$ and $V = B - B_0$:

$$\begin{aligned}\dot{U} &= D\nabla^2 U + f_2 D E\frac{\partial U}{\partial x} + a_{10}U + a_{01}V\\ &\quad + \frac{1}{2}(a_{20}U^2 + 2a_{11}UV + a_{02}V^2)\\ &\quad + \frac{1}{6}(a_{30}U^3 + 3a_{21}U^2V + 3a_{12}UV^2 + a_{03}V^3),\end{aligned} \tag{5}$$

$$\begin{aligned}\dot{V} &= D\nabla^2 V + f_1 E\frac{\partial V}{\partial x} + b_{10}U + b_{01}V\\ &\quad + \frac{1}{2}(b_{20}U^2 + 2b_{11}UV + b_{02}V^2)\\ &\quad + \frac{1}{6}(b_{30}U^3 + 3b_{21}U^2V + 3b_{12}UV^2 + b_{03}V^3),\end{aligned} \tag{6}$$

where $a_{nm} = \frac{\partial^{m+n} f}{\partial^n A\partial^m B}|_{A_0,B_0}$ and $b_{nm} = \frac{\partial^{m+n} g}{\partial^n A\partial^m B}|_{A_0,B_0}$.

To construct a Galerkin model, we need to select the modes. We imagine the substrate to extend from $-L$ to L in the x-direction and to be of infinite extent in the y-direction. The requirement of no current at $x = \pm L$ forces the wave number selection $k_1 = \frac{\pi}{2L}$ for the lowest mode. The optimal wave number $k_2 = k$ in the y-direction will be forced by the instability of the trivial state $U = V = 0$. The simplest model will involve the modes:

$$U = A_1(t)e^{i\frac{\pi}{2L}x}\cos ky + A_1^*(t)e^{-i\frac{\pi}{2L}x}\cos ky, \tag{7a}$$

$$V = B_!(t)e^{i\frac{\pi}{2L}x}\cos ky + B_1^*(t)e^{-i\frac{\pi}{2L}x}\cos ky. \tag{7b}$$

Since A_1 and B_1 are complex variables, we will get a four dimensional when Eqs. (7a) and (7b) are inserted in Eqs. (5) and (6) and the terms in $e^{i\frac{\pi}{2L}x}\cos ky$ equated on either sides give

$$\begin{aligned}\dot{A}_1 &= -D\left(\frac{\pi^2}{4L^2} + k^2\right)A_1 + i\frac{\pi}{2L}Df_2EA_1 + a_{10}A_1 + a_{01}B_1\\ &\quad + \frac{a_{30}}{2}|A_1|^2A_1 + \frac{a_{21}}{2}A_1^2B_1^* + a_{21}|A_1|^2B_1\\ &\quad + \frac{a_{12}}{2}B_1^2A_1^* + a_{12}|B_1|^2A_1 + \frac{a_{03}}{2}|B_1|^2B_1,\end{aligned} \tag{8a}$$

$$\begin{aligned}\dot{B}_1 &= -\left(\frac{\pi^2}{4L^2} + k^2\right)B_1 + i\frac{\pi}{2L}f_1EB_1 + b_{10}B_1 + b_{01}A_1\\ &\quad + \frac{b_{30}}{2}|B_1|^2B_1 + \frac{b_{21}}{2}B_1^2A_1^* + b_{21}|B_1|^2A_1\\ &\quad + \frac{b_{12}}{2}A_1^2B_1^* + b_{12}|A_1|^2B_1 + \frac{b_{03}}{2}|A_1|^2A_1.\end{aligned} \tag{8b}$$

We are interested in the fixed point $A_1 = B_1 = 0$ and carrying out a linear stability analysis, we find that the eigen values are given by

$$\lambda^2 - \lambda(\alpha_1 + i\alpha_2) + \beta_1 + i\beta_2 = 0, \tag{9}$$

where

$$\begin{aligned}\alpha_1 &= a_{10} + b_{01} - (1 + D)\left(k^2 + \frac{\pi^2}{4L^2}\right)\\ &= T - (1 + D)K^2,\\ \alpha_2 &= -\frac{\pi}{2L}E(f_1 + f_2D),\\ \beta_1 &= DK^4 - K^2(a_{10} + Db_{01}) + \triangle - \frac{\pi^2}{4L^2}DE^2f_1f_2,\\ \beta_2 &= -\frac{\pi}{2L}E[f_1a_{10} + f_2Db_{01} - (f_1 + f_2)DK^2],\end{aligned} \tag{10}$$

with $\triangle = a_{10}b_{01} - a_{01}b_{10}$, $T = a_{10} + b_{01}$ and $K^2 = \frac{\pi^2}{4L^2} + k^2$.

It should be noted that for our four-dimensional dynamical system, other two eigen values are just the complex conjugates of the roots found from Eq. (9). For instability to occur at least one of the eigen values must have a non negative real part. This fixes the instability condition as

$$\beta_2^2 \geq \alpha_1(\alpha_1\beta_1 + \alpha_2\beta_2). \tag{11}$$

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As the equality in Eq. (11) is satisfied, one root of Eq. (9) will cross the imaginary axis leading to a standard Hopf bifurcation. The frequency will be either of $\alpha_2 \pm \gamma_2$ (depending on the system parameters), where

$$\gamma_2 = \frac{1}{\sqrt{2}}\{[[\alpha_1^2 - \alpha_2^2 - 4\beta_1]^2 + 4(\alpha_1\alpha_2 - 2\beta_2)^2]^{1/2} - (\alpha_1^2 - \alpha_2^2 - 4\beta)\}^{1/2}. \tag{12}$$

In terms of the system parameters the condition of Eq. (11) becomes

$$[T - (1+D)K^2]^2[K^2(a_{10} + Db_{01}) - b_{01} - DK^4] - (f_1 - Df_2)^2[a_{10}b_{01} - K^2(a_{10} + Db_{01}) + DK^4]\frac{\pi^2}{4L^2}E^2 > 0. \tag{13}$$

The situation for $D \ll 1$ has been extensively analysed (diffusion driven instability). Here we focus on the drift driven instability for D of $O(1)$. We will set $D = 1$ for rest of our analysis.

To understand the existence of the double Hopf bifurcation, we need to look at the second Hopf point. It is important to write out the roots of Eq. (9) as

$$2\lambda = \alpha_1 + i\alpha_2 \pm \sqrt{\alpha_1^2 - \alpha_2^2 - 4\beta_1 + i(2\alpha_1\alpha_2 - 4\beta_2)}, \\ = \alpha_1 + i\alpha_2 \pm (\gamma_1 + i\gamma_2). \tag{14}$$

One Hopf bifurcation is obtained for $\alpha_1 = \gamma_1$, the other for $\alpha_1 = -\gamma_1$. The two Hopf bifurcations meet at $\alpha_1 = \gamma_1 = 0$ which is the condition for double Hopf bifurcation.

From Eq. (10) we immediately see that

$$T = 2K^2. \tag{15}$$

It is to be understood that Eq. (15) is to be used in conjunction with Eq. (11). At the double Hopf point the two frequencies are $\alpha_2 \pm \sqrt{\alpha_2^2 + 4\beta_1}$. This sets up the possibility of a quasi periodic state at the bifurcation point.

Having found the condition for the Hopf bifurcations and double Hopf bifurcation, we now examine the role of wave numbers and hence the box size which will have very important bearing on our discussions. To establish this we first write the threshold value of the electric field for the onset of the Hopf bifurcation at $D = 1$ (this is the case of interest for us, far removed from the much studied Turing patterns for $D \ll 1$). From Eq. (13) we have

$$\pi^2\frac{E_c^2}{4L^2} = \frac{(T - 2K^2)^2(TK^2 - K^4 - \triangle)}{[(f_1 - f_2)^2(a_{10}b_{01} - TK^2 + K^4)]}, \tag{16}$$

where we have explicitly written k_1 as $\pi/2L$. In the complete absence of diffusion the threshold is driven clearly by the differential mobility as noted by Rovinsky and Menzinger.[9] With the diffusion in place as it must be, we need to differentiate between $T < 0$ and $T > 0$ situations. We keep $\triangle > 0$ to prevent the possibility of a saddle in the drift and diffusion free case.

III. RESULTS AND AMPLITUDE EQUATIONS

We begin by exploring the linear stability results of Sec. II for two different situations, $T > 0$ and $T < 0$.

A. $T < 0$

In this case in the absence of diffusion and drift the origin is a stable focus. At $D = 1$, diffusion cannot change the nature of this point. It is left to the electric field to trigger an instability which it can only if the denominator in Eq. (16) is negative. With $a_{10}b_{01} < 0$, this can happen only if $k^2 < k_0^2$, where $k_0^4 - Tk_0^2 + a_{10}b_{01} = 0$. This immediately means that if the box size $L < L_0$ where $L_0 = \pi/2k_0$, we cannot have an instability because the box is not long enough in the direction of the field to sustain the pattern. We now consider the specific case of the Koch Meinhardt (KM) model[20,30] for which $f(A, B) = A^2/B - A + \sigma$ and $g(A, B) = \mu(A^2 - B)$. We can calculate all the a_{ij}, b_{ij} from this and get $T = [(1 - \sigma)/(1 + \sigma) - \mu]$ and $\triangle = \mu$. With $T < 0$ in mind we choose $\sigma = 1/2$ and $\mu = 1/2$. The critical wave number works out to be $\sqrt{[\frac{1-\sigma}{1+\sigma}]} = \sqrt{1/3}$. According to our calculation there can be no instability for $k > 1/\sqrt{3}$. For $k = 1/\sqrt{6}$, the $(\frac{\pi E_c}{2L})^2 = 5/4$. We numerically solved Eqs. (8a) and (8b) and found as expected that for k set at $1/\sqrt{2}$, there is no instability however high the field. This is shown in Figs. 1 and 2. With k set at $1/\sqrt{6}$,we see that the origin is stable for $E < E_c$ (Fig. 3) and for $E > E_c$ (Fig. 4) the origin is clearly unstable and eventually the system settles down in a periodic state with the frequency given by $\alpha_2 - \gamma_2$ as expected from the Hopf bifurcation. Thus, the numeric bears out the expectations.

B. $T > 0$

In this case in the absence of diffusion and drift the origin is an unstable focus. With the help of diffusion the unstable focus can be stabilized with help from the system size. The

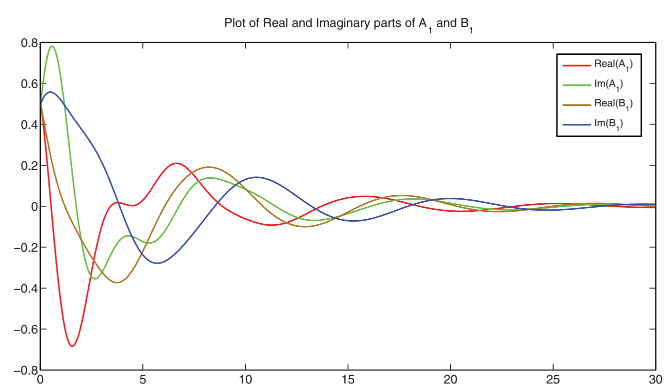

FIG. 1. Plot of the real and imaginary parts of the modes A_1 and B_1 (KM model) for $E = 4$ and $k^2 = \frac{1}{2}$. It should be noted that $E < E_c$ and $k > k_0$ are both preventing pattern formation. As predicted the modes decay to zero.

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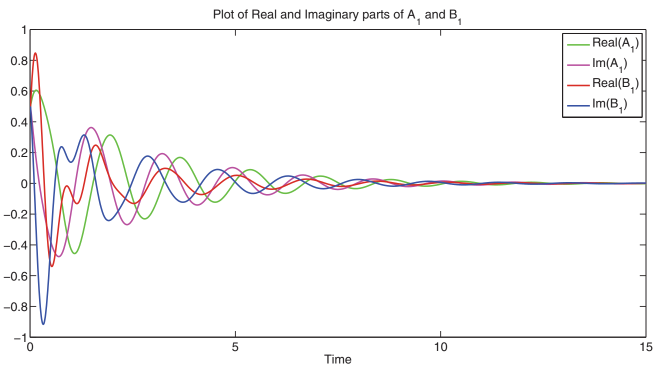

FIG. 2. Plot of the real and imaginary parts of the modes A_1 and B_1 (KM model) for $E = 10$ and $k^2 = 1/2$. Now E is above threshold but since system size is not large enough there should be no pattern formation. We see the modes decaying to 0.

stabilization will occur ($D = 1$) for $k^2 > k_c^2 = T/2$ and can be ensured if the box size L is smaller than $\pi/2k_c$. This allows us to write the roots of Eq. (9) as

$$\lambda = k_c^2 - k^2 - ik_1 E\frac{(f_1+f_2)}{2} \pm\sqrt{k_c^4 - \triangle - \frac{k_1^2E^2(f_1-f_2)^2}{4} + ik_1\frac{E(f_1-f_2)(a_{10}-b_{01})}{2}}. \tag{17}$$

The instability sets in as a Hopf bifurcation when $k = k_c$ and $E = 0$ with $\triangle > k_c^4$. At this point lambda consists of two pairs of imaginary roots which for $E > 0$ acquires positive real parts and thus gives a double Hopf instability leading to a quasi periodic state whose existence, however, can only be guaranteed by the nonlinear analysis to be described below. These features are to be easily found in the Lengeyl Epstein (LE) model[31] where $f(A, B) = \sigma b[B - \frac{AB}{(1+B^2)}]$ and $g(A, B) = a - B - 4\frac{AB}{(1+B^2)}$ with a, b, and σ as positive constants. We list the constants appearing in Eqs. (8a) and (8b) for this model below (we use the notation $c = a/5$ and

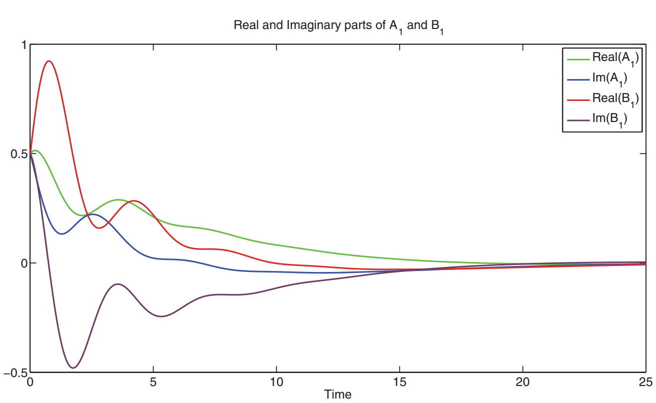

FIG. 3. Plot of the real and imaginary parts of the modes A_1 and B_1 (KM model) for $E = 1$ (below threshold) and $k^2 = \frac{1}{6}$ (allows pattern formation). As expected the modes decay to zero.

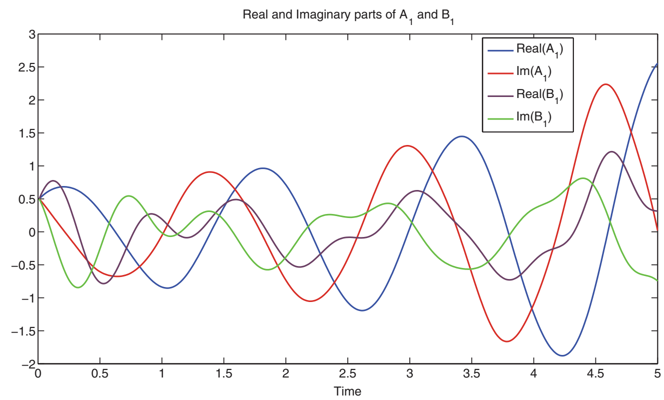

FIG. 4. Plot of the real and imaginary parts of the modes A_1 and B_1 (KM model) for $E = 10$ (above threshold) and $k^2 = \frac{1}{6}$ (allows pattern formation). We see growing modes as expected.

$f = 1 + c^2$):

$$a_{10} = \frac{-\sigma bc}{1+c^2}, \quad a_{01} = \frac{2\sigma b}{f},$$

$$b_{10} = \frac{-4c}{f}, \quad b_{01} = \frac{3c^2-5}{f},$$

$$a_{30} = a_{21} = 0, \quad a_{12} = 2\sigma bc\frac{4-f}{f^2},$$

$$a_{03} = 6\sigma b\frac{1+c^4-46c^2}{f^3},$$

$$b_{30} = b_{21} = 0, \quad b_{12} = 8c\frac{(3c^2-1)}{f^3},$$

$$b_{03} = -8\frac{(9c^4+1-14c^2)}{f^3}.$$

We have not considered the KM model here because all the nonlinear terms in the B_1 equation vanish for this model. We expect the pattern to be quasi periodic in the Lengeyl Epstein model as the following non linear analysis shows.

To investigate the effect of nonlinear terms, we need to proceed by constructing an amplitude equation for the amplitude of the periodic state. To do this, we write the system of Eqs. (8a) and (8b) as ($D = 1$):

$$\dot{A}_1 = \alpha_{11}A_1 + \alpha_{12}B_1 + N_1, \tag{18a}$$

$$\dot{B}_1 = \alpha_{21}A_1 + \alpha_{22}B_1 + N_2, \tag{18b}$$

where $\alpha_{11} = -K^2 + \frac{i\pi}{2L}f_2E + a_{10}$, $\alpha_{12} = a_{01}$, $\alpha_{21} = b_{10}$, $\alpha_{22} = -K^2 + \frac{i\pi f_1}{2L}E + b_{01}$, and N_1 and N_2 are the nonlinear terms of Eqs. (8a) and (8b). We combine the two equations above to write

$$\left[\frac{d^2}{dt^2} - (\alpha_{11}+\alpha_{22})\frac{d}{dt} + \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}\right]A_1 = \dot{N}_1 + \alpha_{12}N_2 - \alpha_{22}N_1. \tag{19}$$

The operator on the left-hand side of Eq. (19) depends on E and $E = E_c$ (given by Eq. (16)), it factors into two roots, $\iota\omega$ and $-\beta$, where $\omega = \alpha_2 \pm \gamma_2$ and $\beta = 2\alpha_1 + \iota(\alpha_2 \mp \gamma_2)$, with

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$\alpha_1 > 0$. We can consequently write

$$\left(\frac{d}{dt} - \iota\omega\right)\left(\frac{d}{dt} + \beta\right)A_1 = \triangle E\left(\frac{\pi^2}{2L^2}E_c f_1 f_2 - \frac{\iota\pi}{2L}(f_1 a_{10} + f_2 b_{01} - K^2(f_1 + f_2))\right)A_1 + \frac{\iota\pi}{2L}(f_1 + f_2)\triangle E\dot{A}_1 + \dot{N}_1 + \alpha_{12}N_2 - \alpha_{22}N_1. \quad (20)$$

To look for periodic states, we seek solutions of Eq. (20) which have the structure

$$A_1 = F(t)e^{\iota\omega t} + G(t)e^{-\iota\omega t}, \quad (21)$$

where $F(t)$ and $G(t)$ are slowly varying functions of time. It is expected that if a periodic state exists then the dynamics of F and G will evolve to stable fixed points. Thus, we need to obtain the dynamics of F and G assuming that they are slowly varying functions. It should be noted that a fixed point $F \neq 0$, $G = 0$ or $F = 0$, $G \neq 0$ will correspond to a travelling wave and $F = G$ will correspond to a standing wave. The various methods of obtaining the evaluation equations for $F(t)$ and $G(t)$ are well documented. They involve perturbation theory and removal of secular terms (i.e., terms of the form $e^{\pm\iota\omega t}$) at each order of perturbation theory. Thus, the calculation to the lowest order involves Eq. (21) and the corresponding $B_1(t)$ given by

$$B_1(t) = (\alpha_{12})^{-1}[(\iota\omega - \alpha_{11})Fe^{\iota\omega t} - (\iota\omega + \alpha_{11})Ge^{-\iota\omega t}], \quad (22)$$

to evaluate the non linear terms in Eq. (20) and extracting the $e^{\pm\iota\omega t}$ parts. Straightforward algebra leads to the following amplitude equations:

$$\dot{F} = \triangle E\left(\left[\frac{\pi^2}{2L^2}E_c f_1 f_2 - \frac{\pi\omega}{2L}(f_1 + f_2)\right]\right)F - \triangle E\left(\frac{\iota\pi}{2L}[f_1 a_{10} + f_2 b_{01} - K^2(f_1 + f_2)]\right)F - (\mu_1 + \iota\mu_2)|F|^2F - (\nu_1 + \nu_2)|G|^2F, \quad (23)$$

$$\dot{G} = \triangle E\left(\left[\frac{\pi^2}{2L^2}E_c f_1 f_2 + \frac{\pi\omega}{2L}(f_1 + f_2)\right]\right)G - \triangle E\left(\frac{\iota\omega}{2L}[f_1 a_{10} + f_2 b_{01} - K^2(f_1 + f_2)]\right)G - (\mu_1 + \iota\mu_2)|G|^2G - (\nu_1 + \nu_2)|F|^2G. \quad (24)$$

For $\mu_{1,2}$ and $\nu_{1,2}$ we obtain long complicated expressions in terms of a_{mn} and b_{mn} which by themselves are not illuminating. The important thing to note is that the travelling wave fixed points ($F = \frac{\triangle E[\frac{\pi^2}{2L^2}E_c f_1 f_2 - \frac{\pi\omega}{2L}(f_1+f_2)]}{\mu}$, $G = 0$) and ($F = 0$, $G = \frac{\triangle E[\frac{\pi^2}{2L^2}E_c f_1 f_2 + \frac{\pi\omega}{2L}(f_1+f_2)]}{\mu}$) exist and are stable, while the standing wave fixed point F = G does not exist yielding contradictory values for Eqs. (23) and (24). This establishes that the Hopf bifurcation does lead to travelling wave states.

The last issue to address is what would the situation be near a double Hopf point. In this case the right-hand side of Eq. (20) would have the structure $(\frac{d}{dt} - \iota\omega_1)(\frac{d}{dt} - \iota\omega_2)A$ and we would be looking for a long time solution which has the structure

$$A_1(t) = [F_1(t)e^{\iota\omega_1 t} + F_2(t)e^{\iota\omega_2 t}], \quad (25)$$

where F_1 and F_2 are slowly varying functions which will satisfy the amplitude equations:

$$\dot{F}_1 = \triangle E\left(\left[\frac{\pi^2}{2L^2}E_c f_1 f_2 - \frac{\pi\omega}{2L}(f_1 + f_2)\right]\right)F_1 - \triangle E\left(\frac{\iota\pi}{2L}[f_1 a_{10} + f_2 b_{01} - K^2(f_1 + f_2)]\right)F_1 - (\mu_1 + \iota\mu_2)|F_1|^2F_1 - (\nu_1 + \nu_2)|F_1|^2F_1, \quad (26)$$

$$\dot{F}_2 = \triangle E\left(\left[\frac{\pi^2}{2L^2}E_c f_1 f_2 + \frac{\pi\omega}{2L}(f_1 + f_2)\right]\right)F_2 - \triangle E\left(\frac{\iota\omega}{2L}[f_1 a_{10} + f_2 b_{01} - K^2(f_1 + f_2)]\right)F_2 - (\mu_1 + \iota\mu_2)|F_2|^2F_2 - (\nu_1 + \nu_2)|F_1|^2F_2. \quad (27)$$

The fixed point that is relevant is both F_1 and F_2 are non zero, and in that case

$$\mu_1|F_1|^2 + \nu_1|F_2|^2 = \frac{\pi^2}{2L^2}E_c f_1 f_2 - \frac{\pi\omega_1}{2L}(f_1 + f_2),$$

$$\nu_1|F_1|^2 + \mu_1|F_2|^2 = \frac{\pi^2}{2L^2}E_c f_1 f_2 - \frac{\pi\omega_2}{2L}(f_1 + f_2).$$

The constraint of $|F_1|^2$ and $|F_2|^2$ both being positive leads to constraints on the coefficients. We see that

$$|F_1|^2 = \frac{\pi^2}{2L^2}\left(\frac{E_c f_1 f_2}{\mu_1 + \nu_1}\right) - \frac{\pi}{2L}f_1 f_2\left(\frac{\mu_1\omega_1 - \nu_1\omega_2}{\mu_1^2 - \nu_1^2}\right),$$

and

$$|F_2|^2 = \frac{\pi^2}{2L^2}\left(\frac{E_c f_1 f_2}{\mu_1 + \nu_1}\right) + \frac{\pi}{2L}f_1 f_2\left(\frac{\nu_1\omega_1 - \mu_1\omega_2}{\mu_1^2 - \nu_1^2}\right).$$

IV. CONCLUSION

Here we have actually constructed amplitude equations for the Turing-Hopf instability to see whether it is a travelling wave or a standing wave. It is clearly seen that the instability is of the travelling wave variety when there is an advective term present. For a finite size system with an advective term, we can have a double Hopf bifurcation if the system size is such that the trace of the linear instability matrix is positive. For the LE system a case study can be made for sigma, $b = 1$ and $a = 5\sqrt{3}$ whence the trace $T = 0.57$ and we can have a double Hopf bifurcation with $k_c = 0.53$. In this special case the parameters of our model appearing in Eqs. (8a) and (8b) are $a_{30} = a_{12} = a_{21} = b_{30} = b_{21} = 0$, $a_{03} = -12$, $b_{12} = \sqrt{3}$, and $b_{03} = -5$. The negative signs associated with a_{03} and b_{03} ensure the existence of the quasi-periodic state. We thus see that finite size effects in an advective reaction diffusion system can induce quasi periodic travelling wave states near the threshold of the instability through a double Hopf bifurcation.

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